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On the Stability of the Quenched State in Mean-Field Spin-Glass Models

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While the Gibbs states of spin-glass models have been noted to have an erratic dependence on temperature, one may expect the mean over the disorder to produce a continuously varying "quenched state." The assumption of such continuity in temperature implies that in the infinite-volume limit the state is stable under a class of deformations of the Gibbs measure. The condition is satisfied by the Parisi Ansatz, along with an even broader stationarity property. The stability conditions have equivalent expressions as marginal additivity of the quenched free energy. Implications of the continuity assumption include constraints on the overlap distribution, which are expressed as the vanishing of the expectation value for an infinite collection of multi- overlap polynomials. The polynomials can be computed with the aid of a *real*-replica calculation in which the number of replicas is taken to zero.

KEY WORDS: Mean field; spin glass; quenched state; overlap distribution; replicas.

1. INTRODUCTION

We consider here the quenched state of the Sherrington-Kirkpatrick (SK) spin glass model, and discuss some stationarity properties which seem to emerge in the infinite volume limit.

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The SK spin glass model[1] has spin variables $\sigma_i = \pm 1$, $i = 1,..., N$, interacting via the Hamiltonian

$$H_N(\sigma, J) = -\frac{1}{\sqrt{N}} \sum_{1 \leqslant i < j \leqslant N} J_{i,j} \sigma_i \sigma_j \tag{1.1}$$

with $J_{i,j}$ independent normal Gaussian variables.

Sampling repeatedly spin configurations $\sigma^{(l)}$ from the space $\mathscr{S}_N = \{-1, 1\}^N$, distributed independently relative to a common Gibbs state, one obtains a random matrix of overlaps $q_{l,m} = q_{\sigma^{(l)}, \sigma^{(m)}}$, with the overlap defined for pairs of spin configurations σ and σ' as:

$$q_{\sigma, \sigma'} = \frac{1}{N} \sum_i \sigma_i \sigma'_i \tag{1.2}$$

The quenched free energy, at a given temperature $T = (k\beta)^{-1}$, is determined in this model by the joint distribution of such overlaps for arbitrary number of "replicas" (copies of the spin system subject to the same random interaction).

The purported solution of the model via the Parisi ansatz[2] has a number of remarkable stability properties. Our purpose is to discuss some of these, starting from elementary continuity assumptions without assuming the validity of the proposed solution. Some results in a similar direction were previously obtained by F. Guerra.[3]

Following are the main observations presented here.

(1) We identify a stability condition, in the sense of invariance of the quenched state in the thermodynamic limit under a broad class of deformations, which is satisfied by the state corresponding to the Parisi solution.

(2) We show that a restricted version of the stability condition is a consequence of a property which make good physical sense, namely the continuity of the quenched ensemble as function of the temperature.

(3) The restricted stability of the quenched state implies the vanishing of expectation values for a family of overlap multireplica polynomials. This accounts for some of the relations, though far from all, found in the Parisi solution.

(4) We ask whether the stability condition singles the Parisi family of states (the GREM models in the terminology of ref. 4). A considerably simplified version of such a question (concerning the characterization of the the REM states) has a positive answer[5] (in preparation).

2. INVARIANCE OF THE PARISI SOLUTION

Let us introduce the notation employed here for the quenched state, and state a remarkable invariance property of the Parisi solution.

2.1. Notation for Quenched Ensembles

In disordered spin systems we encounter two distinct random structures: the spins, distributed (in equilibrium) according to the Gibbs distribution, and the random couplings (and/or random fields and other parameters) which affect the Gibbs state, making it into a random measure. We denote here by $\langle - \rangle$, and in case of possible confusion by $\langle - \rangle_J$, the expectation value over the spins averaged with respect to the Gibbs state. An average over the couplings is denoted by $Av(-)$. The combined *quenched average* is a double average, denoted below by $E(-)$, over the spins and the disorder (whose probability distribution is not affected [in the quenched case] by the response of the spin system to the random Hamiltonian).

Quantities of interest include:

$$E^{(N)}(q_{1,2}^2) = \frac{1}{N^2} \sum_{i,j} Av_N(\langle \sigma_i \sigma_j \rangle_J^2) = Av_N(\langle \sigma_1 \sigma_2 \rangle_J^2) + O\left(\frac{1}{N}\right)$$

and (2.1)

$$\sqrt{N}\, Av_N(\langle \sigma_1 \sigma_2 \rangle\, J_{2,3} \langle \sigma_3 \sigma_1 \rangle)$$

(where the indices on $q_{1,2}$ have different meaning than those on $\sigma_1 \sigma_2$). The second example is seen among other terms in $(\partial/\partial\beta)\, E^{(N)}(q_{1,2}^2)$. In expressions like this the factors $J_{i,j}$ can be integrated out, e.g., through the integration by part formula (for normal Gaussian variables)

$$Av(Jf(J)) = Av\left(\frac{\partial}{\partial J} f(J)\right) \tag{2.2}$$

The expression can then be further reduced to an average of a suitable overlap monomial.

Overlap monomials are functions of the form $\prod_{1 \leq l,m \leq K} q_{\sigma^{(l)},\sigma^{(m)}}^{n_{l,m}}$ defined over the product space $\mathcal{S}^{\otimes K}$, $\mathcal{S}$ being the spin configuration space. Their expectation values over the corresponding product measure on identical copies $\prod_{1 \leq l \leq K} \langle - \rangle_{N,J}^{(l)}$ are denoted by the symbol

$$\left\langle\!\!\left\langle \prod_{1 \leq i,j \leq K} q_{\sigma^{(i)},\sigma^{(j)}}^{n_{i,j}} \right\rangle\!\!\right\rangle_{N,J} \tag{2.3}$$

and the full "quenched average" is denoted by $E(-)$, e.g.:

$$E^{(N)}\left(\prod_{1 \leqslant l, m \leqslant K} q_{l,m}^{n_{l,m}}\right) = Av_N\left(\left\langle\!\!\left\langle \prod_{1 \leqslant i, j \leqslant K} q_{\sigma^{(l)}, \sigma^{(m)}}^{n_{l,m}} \right\rangle\!\!\right\rangle_{N, J}\right) \tag{2.4}$$

We shall often omit the explicit reference to J, to other subscripts such as temperature, as well as to N. However, it should hopefully be clear from the context when do we refer to a finite system and when to the infinite volume limit.

Naturally, we are interested in the limit $N \to \infty$. There is no reason to expect (at low temperatures) convergence of the state $\langle\!\langle - \rangle\!\rangle_J$ at given realization of the random couplings $\{J_{i,j}\}$. However it does not seem unreasonable to expect convergence of the quenched averages of $\langle\!\langle f(\sigma) \rangle\!\rangle$, where $f(*)$ can be any local function of the spins. We note that elementary compactness arguments imply that for any temperature there is a sequence N_k which increases to ∞, for which the following limits exists simultaneously for all the overlap monomials

$$\lim_{k \to \infty} E^{(N_k)}\left(\prod_{1 \leqslant l, m \leqslant K} q_{\sigma^{(l)}, \sigma^{(m)}}^{n_{l,m}}\right) = E\left(\prod_{1 \leqslant l, m \leqslant K} q_{l,m}^{n_{l,m}}\right) \tag{2.5}$$

Our discussion will concern relations among the monomial averages which would be valid in such limits, under a number of assumptions (which include the existence of the limit).

Remark. It is possible to develop a more complete setup for the formulation of the infinite volume limit of states of the SK model, in which the limit of the quenched averages is described in terms of a probability measure on $\mathcal{M}(\mathcal{S})$—the space of probability measures on $\mathcal{S} = \{-1, +1\}^{\mathbb{N}}$ which is the space of configurations of an infinite spin system ($\mathbb{N}$ being the set of natural numbers). The elements of $\mathcal{M}(\mathcal{S})$ are random states incorporating the effects of quenched disorder. We shall, however, not pursue this line here.

We shall invoke an additional element of structure: Gaussian fields $h(\sigma)$, $K(\sigma)$ defined over $\mathcal{S}_N$ with the covariances

$$Av(h(\sigma^{(l)})\, h(\sigma^{(m)})) = q_{l,m} \tag{2.6}$$

$$Av(K(\sigma^{(l)})\, K(\sigma^{(m)})) = q_{l,m}^2 \tag{2.7}$$

$$Av(h(\sigma^{(l)})\, K(\sigma^{(m)})) = 0 \tag{2.8}$$

which are independent of each other and of J, in the sense exemplified by the relation

$$E^{(N)}(e^{\sum_j \lambda_j h(\sigma^{(j)})}) = E^{(N)}(e^{(1/2)\sum_{n,m} \lambda_n \lambda_m q_{n,m}}) \tag{2.9}$$

(Analogous relation—with $q_{i,j}$ replaced by $q_{i,j}^2$, is assumed for K.)

In the SK model ($N < \infty$), a quantity like $h(\sigma)$ appears as the *cavity field* associated with an increase in N, and $K(\sigma)$ can be found as representing the change in the action corresponding to an increase in the temperature. The two can be realized as:

$$h(\sigma) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} J'_i \sigma_i \tag{2.10}$$

and

$$K(\sigma) = \frac{1}{N} \sum_{i<j} J''_{i,j} \sigma_i \sigma_j \tag{2.11}$$

where the J'_i and $J''_{i,j}$ are normal Gaussian variables, independent form each other and from the couplings J appearing in the Hamiltonian. Based on this example, we incorporate the averages over the fields h, K under the symbol $Av(-)$, even when the average is at fixed σ.

2.2. Invariance of the Parisi Solution

The Parisi solution has the property that quenched averages are not affected by the addition to the action of terms of the form $F(K(\sigma), h(\sigma))$, where $F(\cdot, \cdot)$ is any smooth bounded function.

To express the above stated property, let us consider the deformed states

$$\begin{aligned} \langle - \rangle_{N, F(K,h)} &:= \frac{\langle - \exp\{F(K,h)\} \rangle_N}{\langle \exp\{F(K,h)\} \rangle_N} \\ \langle\!\langle - \rangle\!\rangle_{N, F(K,h)} &= \bigotimes_l \langle - \rangle^{(l)}_{N, F(K,h)} \end{aligned} \tag{2.12}$$

and

$$E^{(N)}_{F(K,h)}(-) = Av(\langle\!\langle - \rangle\!\rangle_{N, F(K,h)}) \tag{2.13}$$

and let $E_{F(K,h)}(-)$ represent the corresponding limit, as $N \to \infty$, for the expectation values of overlap monomials (assuming the limit exists).

Claim 2.1. Assuming the validity of the Parisi solution, in the infinite volume limit at any temperature:

$$E_{F(K,h)}\left(\prod_{1\leq l,m\leq K} q_{l,m}^{n_{l,m}}\right)=E\left(\prod_{1\leq l,m\leq K} q_{l,m}^{n_{l,m}}\right) \tag{2.14}$$

where the expectation value functionals are to be interpreted as the $N\to\infty$ limits of expectations of overlap monomials.

We shall not verify this statement here—the reader is invited to do so from the solution which is discussed in ref. 2, references therein, and in ref. 4 —instead we shall discuss the origin and consequences of a somewhat restricted invariance of this kind.

3. CONTINUITY IN THE TEMPERATURE AND STABILITY UNDER DEFORMATION

The broad stability of the quenched state expressed by Eq. (2.14) has not yet been rigorously derived for the SK model. We shall now find that a somewhat restricted version of this condition follows from a natural continuity assumption.

There is a significant difference between the spin-glass and the ferromagnetic spin models in the effect of a change in the temperature on the equilibrium state. Reduction in the temperature amounts to increased coercion towards the low energy states of the system. If the ground state is unique, it is natural to expect the equilibrium state to vary continuously at low temperatures. When there are only few ground states, one may expect some discontinuities (as in the Pirogov-Sinai theory[6]). However, when there is a high multiplicity of competing low energy states the result may be quite different. Indeed it is reported that for a given realization of the random Hamiltonian, the equilibrium state has a very erratic dependence on the temperature. Nevertheless it may seem reasonable to expect that with the average over the disorder, the quenched state might vary continuously with β.

To illuminate the consequences of the continuity assumption, let us note that due to the addition law for independent Gaussian variables, the Gibbs factor determining the equilibrium state at the inverse-temperature $\beta+\Delta\beta$ can be presented as a sum of two independent terms:

$$(\beta+\Delta\beta)\, H(\sigma,J) \stackrel{\mathcal{D}}{=} \beta H(\sigma,J)+\delta(\beta)\, H(\sigma,\tilde{J}) \tag{3.1}$$

where $A \stackrel{\mathscr{D}}{=} B$ means that A and B have equal probability distributions, $\{J, \tilde{J}\}$ are two independent sets of couplings, and

$$\delta(\beta) = \sqrt{2\beta\, \Delta\beta + (\Delta\beta)^2} \tag{3.2}$$

With the action cast in the form Eq. (3.1), the modified state is seen to incorporate the effects of a strong term (of the order of the volume) pulling in some randomly chosen directions, when the main term itself has many competing states. The assumption of the continuity of the quenched state appears now as less obvious, and it should therefore carry some notable consequences.

A related observation can be made by considering the effects of deformation of the state through the addition of a Gaussian field of the type $K(\sigma)$ (Eq. (2.7)). An easy computation based on the fact that

$$H(\sigma, \tilde{J}) \stackrel{\mathscr{D}}{=} \sqrt{N}\, K(\sigma) \tag{3.3}$$

shows that

$$\sqrt{\beta^2 + \frac{\lambda^2}{N}}\, H(\sigma, J) \stackrel{\mathscr{D}}{=} \beta H(\sigma, J) + \lambda K(\sigma) \tag{3.4}$$

i.e., the deformation with a field K is equivalent in a change of the order $O(1/N)$ in the temperature:

$$E_{\lambda K}^{(N,\beta)}\left(\prod q_{i,j}^{n_{i,j}}\right) = E^{(N, \sqrt{\beta^2 + \lambda^2/N})}\left(\prod q_{i,j}^{n_{i,j}}\right) \tag{3.5}$$

where the superscripts refer to the size and the inverse-temperature, the subscript indicates a deformation of the state in the sense of Eq. (2.12) and, as for the Eq. (2.14) the equality is understood when the two measures are restricted to the quantities independent from the deformation variable K.

In the limit $N \to \infty$, the change in the temperature on the right side vanishes. This immediately leads to the following observation.

Proposition 3.1. If a certain temperature range the quenched averages $E^{(N,\beta)}(\prod q_{i,j}^{n_{i,j}})$ are uniformly continuous in β, as $N \to \infty$, and the infinite-volume limit exist for the quenched state, in the sense of Eq. (2.5), then the limit $E^{(\beta)}(\prod q_{i,j}^{n_{i,j}})$ is stable under deformations by $e^{\lambda K}$, i.e.:

$$e^{(\beta)}\left(\prod q_{i,j}^{n_{i,j}}\right) = E_{\lambda K}^{(\beta)}\left(\prod q_{i,j}^{n_{i,j}}\right) \tag{3.6}$$

for all the overlap monomials.

Let us note that the assumptions made above imply also another stationarity principle: the quenched state would be invariant under the deformation induced by $\ln 2\cosh \beta h$. To see that, compare the state of $N+1$ particles with that of N. The trace over the "last" spin yields for expectation values of functions of the "first" N spins

$$E^{(N+,\beta)}\left(\prod q_{i,j}^{n_{i,j}}\right)=E^{(N,\tilde{\beta})}_{\ln 2\cosh \tilde{\beta} h}\left(\prod q_{i,j}^{n_{i,j}}\right) \tag{3.7}$$

where $\tilde{\beta}=\sqrt{N/(N+1)}\,\beta$ and $h(\sigma)$ is a Gaussian field with covariance $q_{\sigma,\sigma'}$. Under the stated assumptions, in the thermodynamical limit the previous relation becomes:

$$E^{(\beta)}\left(\prod q_{i,j}^{n_{i,j}}\right)=E^{(\beta)}_{\ln 2\cosh \beta h}\left(\prod q_{i,j}^{n_{i,j}}\right) \tag{3.8}$$

4. A LOGARITHMIC RELATION EXPRESSING THE STABILITY PROPERTY

The stability condition which follows from the above discussed continuity assumption is indeed found among the properties of the Parisi solution, along with the more sweeping stability under deformations of the more general form, as seen in Eq. (2.14). This invariance property can also be cast in the form of a "logarithmic relation," which expresses an additivity property for the marginal increments in the quenched free energy.

Definition 4.1. We say that a random system, in the quenched state $AV_N(\langle\!\langle - \rangle\!\rangle_N)$, has marginally-additive free energy if for any finite collection of independent Gaussian fields $K^{(1)}, K^{(2)},..., K^{(l)}$ with the covariance Eq. (2.7), and any smooth polynomially bounded functions $F_1, F_2,..., F_l$, the following limits exist and satisfy

$$\lim_{N\to\infty} Av_N \ln\left\langle \exp\left(\sum_{i=1}^{l} F_i(K^{(i)})\right)\right\rangle_N = \lim_{N\to\infty} \sum_{i=1}^{l} Av \ln\langle \exp F_i(K^{(i)})\rangle_N \tag{4.1}$$

where the indices label independent families of fields (not to be confused with replica indices).

Our main observation is that the above marginal additivity of the quenched free energy is equivalent to the *stability* of the quenched state (in the infinite volume limit) expressed by

$$E\left(\prod q_{i,j}^{n_{i,j}}\right) = E_{F(K)}\left(\prod q_{i,j}^{n_{i,j}}\right) \tag{4.2}$$

where F is an arbitrary smooth bounded function.

Let us note that the expectation values of quantities involving any of the above, can be evaluated by first integrating over the extraneous Gaussian variables (K). Using Wick's formula, this integration produces expressions involving the overlaps among arbitrary number of replicas, as in:

$$\begin{aligned} E^{(N)}(K_1 K_2 K_2' K_3') &= Av_N(\langle K\rangle_N \langle KK'\rangle_N \langle K'\rangle_N) \\ &= E^{(N)}(q_{1,2}^2 q_{2,3}^2) \end{aligned} \tag{4.3}$$

where we defined $K_i = K(\sigma^{(i)})$.

Conversely, the averages of polynomials in replica overlaps, as $q_{1,2}^2 q_{2,3}^2$ in the above expression, can be easily expressed through the expectation values of suitable products of independent copies of the K field.

Clearly the stability implies the marginal additivity property (of the free energy). To prove the converse, we need to show that (assuming the two limits exist)

$$\lim_{N\to\infty} Av_N[\langle G(K)\rangle_{N,F(K')}]^n = \lim_{N\to\infty} Av_N[\langle G(K)\rangle_N]^n \tag{4.4}$$

for any integer n and polynomial function G. (The full statement takes a bit more general form–involving products with different functions G for the different copies of the spin system, however by the polarization argument there is no loss of generality in taking the same function G for all the n replicas.) Let us note also that, by an elementary approximation argument, it suffices to prove Eq. (4.4) for bounded functions G.

The logarithmic property (4.1) implies that for all ε

$$\varphi_N(\varepsilon) \equiv Av_N \ln \frac{\langle \exp(\varepsilon G + F)\rangle_N}{\langle \exp(\varepsilon G)\rangle_N \langle \exp(F)\rangle_N} \xrightarrow[N\to\infty]{} 0 \tag{4.5}$$

For bounded G the function $\varphi_N(\varepsilon)$ is analytical in a strip containing the real axis uniformly in N, and the logarithmic property Eq. (4.1) is equivalent to the vanishing of all the quantities $\varphi_N^{(n)}(\varepsilon)|_{\varepsilon=0}$ in the infinite volume limit. By an inductive argument, these conditions imply Eq. (4.4): first we observe that

$$\varphi'_N(\varepsilon)|_0 = Av_N(\langle G\rangle_{N,F}) - Av_N(\langle G\rangle_N) \to 0 \tag{4.6}$$

which is the stability for the first power (one replica). The second derivative gives

$$\begin{aligned}\varphi''_N(\varepsilon)|_0 &= Av_N(\langle G\rangle^2_{N,F}) - Av_N(\langle G^2\rangle_{N,F}) \\ &\quad - Av_N(\langle G\rangle^2_N) + Av_N(\langle G^2\rangle_N) \to 0\end{aligned} \tag{4.7}$$

Thus

$$\begin{aligned}\lim_{N\to\infty} Av_N(\langle G\rangle^2_{N,F}) - Av_N(\langle G\rangle^2_N) &= \lim_{N\to\infty} Av_N(\langle G^2\rangle_{N,F}) - Av_N(\langle G^2\rangle_N) \\ &= 0\end{aligned} \tag{4.8}$$

where the last equality is by the first order equation, Eq. (4.6), applied to the smooth function G^2. Continuing in this fashion one may see that if stability is fulfilled up to power n it is fulfilled for power $n+1$.

Remark 4.2. The truncated expectations (cumulants) of order p are generally defined by

$$\langle -\,;p\rangle^T = \frac{\partial^p}{\partial\lambda^p}\ln\langle\exp(\lambda -)\rangle|_{\lambda=0} \tag{4.9}$$

In these terms the logarithmic relation is equivalent to:

$$\lim_{N\to\infty} Av_N\left\langle \sum_{i=1}^{l} F_i(K^{(i)});\, p\right\rangle_N^T = \lim_{N\to\infty} Av_N \sum_{i=1}^{l} \langle F_i(K^{(i)});\, p\rangle_N^T \tag{4.10}$$

for every integer p. (The implication Eq. (4.1) $\Rightarrow$ Eq. (4.10) is obvious. In the other direction the proof can be based on the analyticity argument indicated above.) It might be noted that an equation like Eq. (4.10) cannot possibly hold without the average Av, unless the Gibbs state is typically supported on a narrow collection of configurations over which the overlap function takes only the value $q_{\sigma,\sigma'} = 1$.

Equations (4.1) and (4.10) have natural counterparts for the more limited stability of the quenched state, expressed by Eq. (3.6). In that case $F_i(K^{(i)})$ need by replaced by $\lambda_i K^{(i)}$, for $i = 2, 3,...$ (thought F_1 may still be left arbitrary.)

It is an interesting open question whether there are states stable in the limited sense which do not fulfill the stronger stability condition.

5. OVERLAP POLYNOMIALS WITH ZERO AVERAGE

We now turn to some of the implications of the stability condition Eq. (3.6) which was shown to follow from the continuity assumption. Since the free energy and its derivatives are determined through the distribution of the overlaps, it is natural to ask what consequences does the stability property have in those terms. As we shall see next, the implications include a family of relations expressed as the vanishing of the expectation value of an infinite collection of overlap polynomials. From a combinatorial point of view, expressions with vanishing expectation are constructed by applying a certain operation to graphs representing overlap polynomials.

We shall use the notation encountered already in Eq. (4.3), where $q_{1,2}$ indicates the overlap between two spin configurations sampled from two different copies of the system, one in replica 1 and the other in the replica 2, subject to the same random interaction. Products of such terms can be represented by labeled graphs, introduced below. The expectation value does not depended on the particular labeling of the different replicas, for instance

$$E(q_{1,2}^2 q_{2,3}^4 q_{1,4}^2) = E(q_{1,2}^2 q_{2,3}^2 q_{3,4}^4) \tag{5.1}$$

where we omit, as in the rest of this section, the finite volume symbol N, unless otherwise specified.

Let now illustrate some consequences of the stability condition starting from the simple monomial

$$Av(\langle K \rangle_{\lambda K'}^2) \tag{5.2}$$

Stability implies the vanishing of all the derivatives of this function of λ. Let us proceede for a moment under the assumption, which is proven in the appendix, that the limit of the derivatives in λ equals the derivative of the limit, which under the stability condition is zero. The first derivative gives

$$2Av_N(\langle K \rangle_{\lambda K'} (\langle KK' \rangle_{\lambda K'} - \langle K \rangle_{\lambda K'} \langle K' \rangle_{\lambda K'})) \tag{5.3}$$

which, at $\lambda = 0$, vanishes for the trivial reason of parity, at $\lambda = 0$. On the other hand, the second derivative yields

$$\begin{aligned} 2Av_N(&\langle KK' \rangle_{\lambda K'}^2 - 4\langle K \rangle_{\lambda K'} \langle K' \rangle_{\lambda K'} \langle KK' \rangle_{\lambda K'} + 3\langle K \rangle_{\lambda K'}^2 \langle K' \rangle_{\lambda K'}^2 \\ &+ \langle K \rangle_{\lambda K'} \langle KK'^2 \rangle_{\lambda K'} - \langle K \rangle_{\lambda K'}^2 \langle K'^2 \rangle_{\lambda K'}) \end{aligned} \tag{5.4}$$

At $\lambda = 0$ the last two terms cancel and the expression reduces to:

$$2E^{(N)}(q_{1,2}^4 - 8q_{1,2}^2 q_{2,3}^2 + 6q_{1,2}^2 q_{3,4}^2) \tag{5.5}$$

If the quenched state is stable in the sense of Eq. (3.5) the above expression tends to zero in the thermodynamic limit.

As was mentioned already, the stability property is satisfied by the Parisi solution, and hence this relation, as well as those of higher order derived below, are satisfied there. The particular case of vanishing of Eq. (5.5) was recently derived (for almost every *beta*) without any assumptions in ref. 3. One may also note that the vanishing of Eq. (5.5) is also the lowest non-trivial identity of those listed in Eq. (4.10) (corresponding to $p = 4$, $F(K) \equiv K$).

Let us present now a systematic approach for the derivation of other such relations.

One may use a graphical representation in which a monomial of the form $q_{1,2}^2 q_{2,3}$ is identified with a graph whose vertices are the replica indices $\{1, 2, 3\}$ and the edges correspond to the overlaps, $q_{i,j}$. Such a graph will be indicated by the symbol $(1, 2)^2 (2, 3)$. Furthermore, we shall consider also products involving an additional Gaussian field (K). The graphical representation of that factor is a half-edge, represented by a singleton. I.e., $(1, 2)(2)$ and $(1, 2)(3)$ correspond to $q_{1,2}K_2$ and $q_{1,2}K_3$.

We shall use a product "$\cdot$" which acts in the space of graphs as *composition* combined with *contraction*, where possible, of the two unpaired legs. The notion may be clarified by the following examples:

$$(1, 2) \cdot (1, 2)(2) = (1, 2)^2 (2) \tag{5.6}$$

$$(1, 2)(3) \cdot (4) = (1, 2)(3, 4) \tag{5.7}$$

Terms of the form $(1, 1)$ can be omitted, since in our case $q_{1,1} = 1$.

The above product turns out to be commutative but not associative. The *order* of a graph is defined as $2 \times$ *number of edges*, with half-edges counting as 1/2. Let W_k denote the space of formal linear combination of graphs of order k. For $G \in W_k$ we denote by Q_G the corresponding element of the overlap algebra.

We define δ: $W_k \to W_{k+1}$ as the linear operator which acts on single graphs by

$$\delta G = \sum_{v \in V(G)} \delta_v G \tag{5.8}$$

where $V(G)$ is the set of vertices of G, and

$$\delta_v G = G \cdot (v) - G \cdot (\tilde{v}) \tag{5.9}$$

where $\tilde{v}$ is a new vertex not belonging to G. E.g., $\delta_1(1,2) = (1,2)(1) - (1,2)(3)$, $\delta_1(1,2)(1) = (1,2)(1) \cdot (1) - (1,2)(1) \cdot (3) = (1,2) - (1,2)(1,3)$.

Following is the pertinent observation.

Proposition 5.1. For any measure of the type $E^{(N)}(-) = Av_N(\langle\!\langle - \rangle\!\rangle_N)$ and a deformation defined in (3.6)

$$\frac{\partial^2}{\partial \lambda^2} E^{(N)}_{\lambda K}(Q_G)|_{\lambda=0} = E^{(N)}(Q_{\delta^2 G}) \tag{5.10}$$

for all the elements Q_G of the overlap algebra and every N.

The proof of Proposition 5.1 is straightforward. The operation δ is the graphical counterpart of the usual derivative with respect to the parameter λ in the Boltzmann weight (where it appears in λK). Such a derivative produces a *truncated correlation* expressed in the rule (5.9). The (5.8) is nothing but the Leibnitz rule for derivative of products. The first differentiation produces a sum of monomials, each containing an unpaired centered Gaussian variable (K) of zero mean. The second derivative produces another unpaired variable, which is contracted with the previous one via the Wick rule. (This contraction motivates the product introduced above.)

Proposition 5.2. If in a certain temperature range the quenched averages $E^{(N,\beta)}(Q)$ are uniformly continuous in β, as $N \to \infty$, and the infinite-volume limit exist for the quenched state, in the sense of Eq. (2.5), then

$$E(Q_{\delta^2 G}) = 0 \tag{5.11}$$

for every element of the overlap algebra.

The uniform continuity in β implies the stability for deformation Eq. (3.6) which means, in particular, that the limit is a constant in λ. To prove the theorem we have to show that we can interchange the thermodynamical lmit with the repeated differentiation w.r.t. λ, in Eq. (5.10). This is shown to be true in the appendix, through uniform (in N) bounds on the kth derivatives of expectations of overlap monomials.

Following is a related statement which yields a somewhat stronger conclusion (suggesting a numerical test), which is derived under a stronger assumption.

Proposition 5.3. In the SK model, at finite values N,

$$\frac{\partial}{\beta\,\partial\beta} E^{(N)}(Q_G) = N \frac{\partial^2}{\partial\lambda^2} E^{(N)}_{\lambda K}(Q_G)|_{\lambda=0} = N E^{(N)}(Q_{\delta^2 G}) \qquad (5.12)$$

for every element of the overlap algebra and every N. In particular if in a certain range of β the quantities $(\partial/\beta\,\partial\beta)\, E^{(N)}(Q_G)$ are uniformly bounded in N, and the thermodynamic limit exist in the sense of Eq. (2.5), then for all the elements Q_G of the overlap algebra:

$$E(Q_{\delta^2 G}) = O(1/N) \qquad (5.13)$$

The proof of the first equality can be obtained computing the second derivative with respect to λ of equation Eq. (3.5) at $\lambda = 0$. The second equality is Eq. (5.10).

Analogous statements hold for other mean field spin glass models, with the p-spin interaction Hamiltonian[7]

$$H(\sigma, J) = - \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant N} J_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p} \qquad (5.14)$$

where $J_{i_1,\ldots,i_p}$ are Gaussian variables, rescaled so that the Hamiltonian covariance is

$$Av H(\sigma, J)\, H(\sigma', J) = N q^p_{\sigma,\sigma'} \qquad (5.15)$$

(Under the above calling, H reaches values of order N. The corresponding choice for the field K is Gaussian with the covariance $Av(K(\sigma)\, K(\sigma')) = q^p_{\sigma,\sigma'}$. The definition of δ is unchanged.

6. COMPUTATION WITH REAL REPLICAS

In this section we give several characterizations of the overlap polynomials of the form $Q_{\delta^2 G}$ for which we prove that under certain conditions they have zero mean.

The main result is a formula which permits to compute the polynomials from a quadratic expression in the number r of *real*-replicas, evaluated at $r = 0$. To state it, let $M_r = \sum_{i \neq j = 1}^{r} q^2_{i,j}$, for all integers $r \geqslant 1$. Let $E(-)$ be an expectation value functional, on the algebra of overlaps, which depends only on the graph structure of the overlap monomials (i.e., is independent of the choice of labels). Then the quantity $E(Q_G M_r)$ is quadratic in r. In the following proposition, we refer to the *polynomial extension* of this function to all real r.

Proposition 6.1. For any expectation value functional $E(-)$, as above,

$$E(Q_{\delta^2 G}) = E(Q_G M_r)|_{r=0} \tag{6.1}$$

where the quantity $E(Q_G M_r)$ is first computed for r large enough so that all the indices appearing in Q_G do appear also in M_r and $r > |G| + 1$.

To illustrate the statement, let us take: $G = q_{1,2}^2$. In this case, the left side of Eq. (6.1) is given by Eq. (5.5) and the right side is:

$$E(q_{1,2}^2 M_r) = 2E(q_{1,2}^4) + 4(r-2)\, E(q_{1,2}^2 q_{2,3}^2) + (r-2)(r-3)\, E(q_{1,2}^2 q_{3,4}^2) \tag{6.2}$$

The two coincide at $r = 0$ (defined by polynomial extension).

The proof proceeds through the explicit computation of the left and right sides of Eq. (6.1).

Lemma 6.2. If the number of vertices in G is l then

$$\delta^2 G = \sum_{v \neq v'} G \cdot (v, v') - 2l \sum_v G \cdot (v, \tilde{v}) + l(l+1)\, G \cdot (\tilde{v}, \tilde{v}') \tag{6.3}$$

where v and v' are summed over the set of vertices of G and $\tilde{v}$ and $\tilde{v}'$ denote a pair of added vertices.

This is a rather explicit expression for the polynomials corresponding to a given graph G. Two examples are:

$$\begin{aligned}\delta^2(1,2)(3,4) = {} & 4(1,2)^2\,(3,4) + 8(1,2)(2,3)(3,4) \\ & - 32(1,2)(2,3)(4,5) + 20(1,2)(3,4)(5,6)\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\delta^2(1,2)(2,3) = {} & 4(1,2)^2\,(2,3) + 2(1,2)(2,3)(3,1) - 12(1,2)(2,3)(3,4) \\ & + 12(1,2)(2,3)(4,5) - 6(1,2)(2,3)(2,4)\end{aligned} \tag{6.5}$$

Remark. In the above example $\delta^2 G$ is a polynomial expression with integer coefficients whose sum is zero. That property is shared by $\delta^2 G$ of arbitrary monomials G.

To prove the lemma we note that by the definition of δ

$$\delta G = \sum_v G \cdot (v) - lG \cdot (\tilde{v}) \tag{6.6}$$

Applying this rule twice

$$\delta^2 G = \sum_{v, v'} G\cdot(v, v') - l \sum_{v} G\cdot(v, \tilde{v}) - l \sum_{v'} G\cdot(v', \tilde{v}) \\ - l \sum_{v'} G\cdot(\tilde{v}, \tilde{v}) + l(l+1)\, G\cdot(\tilde{v}, \tilde{v}') \tag{6.7}$$

which coincides with Eq. (6.3) since $(v, v) = 1$ for every v.

Lemma 6.3. If all the replica indices appearing in Q_G are contained in the entries of the matrix M_r and $r > l+1$ then

$$E(Q_G M_r) = \sum_{\substack{v, v' \in (G), \\ v \neq v'}} E(Q_{G\cdot(v, v')}) + 2(r-l) \sum_{v \in V(G)} E(Q_{G\cdot(v, \tilde{v})}) \\ + (r-l)(r-l-1)\, E_{G\cdot(\tilde{v}, \tilde{v}')}) \tag{6.8}$$

This formula is an elementary consequence of the fact that the measure $E(-)$ depends only on the isomorphism type of the graph associated to a given overlap monomial. The first sum on the right side of Eq. (6.8) corresponds to those overlap terms in M_r which involve only the replicas appearing in G, the other two sums are split according to whether the number of vertices not in G is 1 or 2.

The two previous lemmas prove Proposition 6.1.

7. COMMENTS

We have seen that elementary continuity assumptions on the quenched state, imply a stability property for the infinite volume limit of the SK (and other mean-field) spin-glass models. A particular implication is the vanishing of the expectation values of certain multi-overlap polynomials, which form an infinite dimensional family. We also saw a related condition expressed through an explicit decay rate for the expectation values of suitable quantities.

These observations are consistent with the Parisi theory. However, the family of identities discussed here does not yet permit the reconstruction of the joint probability distribution from that of a single overlap, as is the case under the Parisi Ansatz.

It has been pointed out that within the replica-symmetry-breaking approach the vanishing of the expectation values of the polynomials discussed here ($\delta^2 G$) requires only the so called "replica-equivalence" assumption, which says that in the matrix Q (defined in ref. 2) each row is a permutation of any other. We thank I. Kondor and M. Mezard for calling our attention to this point. See ref. 8 for a recent account on replica-equivalence.

An interesting question is whether the stability property is the stationarity condition for some variational principle. This is related to the main question which emerges at this point, which is whether stability implies the GREM state structure.(4) We study a restricted version of this question in a separate paper.(5)

APPENDIX

In this appendix we show that the limit $N \to \infty$ can be interchanged with the differentiaion, in formula (5.10). This is seen in two steps. The first is a general criterion.

Theorem 1.1. Let $F_n(\lambda)$ be a sequence of functions defined over the interval $\lambda \in [-1, 1]$, which:

(a) converge pointwise:

$$F_n(\lambda) \xrightarrow[n \to \infty]{} G(\lambda) \qquad \text{for all} \quad \lambda \in [-1, 1] \tag{A.1}$$

(b) have uniformly bounded derivatives up to order $m+1$, i.e. satisfy

$$\left| \frac{d^k}{d\lambda^k} F_n(\lambda) \right| \leqslant \text{Const} \tag{A.2}$$

for all $\lambda \in [-1, 1]$ and $k = 1,..., m+1$.

Then for $k = 1,..., m$ the derivatives also converge (pointwise and uniformly), the limit is differentiable, and

$$\frac{d^k}{d\lambda^k} F_n(\lambda) \xrightarrow[n \to \infty]{} \frac{d^k}{dx^k} G(\lambda) \tag{A.3}$$

We omit the proof of this basic criterion. (For $m = 1$ it can be proven by using uniform approximations for $(d/d\lambda)\, F_n(\lambda)$ in terms of the differences $[F_n(\lambda + \varepsilon) - F_n(\lambda)]/\varepsilon$, and the rest is by induction.)

The stability stated in Proposition 3.1 amounts to an (a) type condition for $F_n(\lambda) = E^{(N)}_{\lambda K}(Q)$ where Q is any overlap monomial. The limit is a constant function. The above principle will allow us to conclude that under the assumption of Proposition (3.1) $E(Q_{\delta^2 G}) = 0$, as soon as we show that the derivatives are uniformly bounded (i.e., establish condition (b)). Following is a detailed version of that statement.

Proposition. Let Q be an overlap monomial which involves r replicas. Then, for any $N < \infty$,

$$\left| \frac{d^k}{d\lambda^k} E^{(N)}_{\lambda K}(Q) \right| \leqslant 2r^k (k!)^2 \, e^{(1+\lambda)^2 + \lambda^2} \tag{A.4}$$

Proof. The expectation value of Q are expressed as $E^{(N)}_{\lambda K}(Q) = Av_N \langle\langle Q \rangle\rangle_{N, \lambda K}$ with a finite product $\langle\langle - \rangle\rangle_{\lambda K} = \bigotimes_{i=1,\dots,r} \langle - \rangle^{(i)}_{\lambda K}$. Using standard formula for the derivative, standard bounds on the truncated correlations, and the fact that $|Q| \leqslant 1$

$$\begin{aligned} \left| \frac{d^k}{d\lambda^k} E^{(N)}_{\lambda K}(Q) \right| &= \left| v_N \left(\left\langle\!\!\left\langle Q; \sum_{i=1}^{r} K(\sigma_i); \dots; \sum_{i=1}^{r} K(\sigma_i) \right\rangle\!\!\right\rangle_{N, \lambda K} \right) \right| \\ &\leqslant k! \, Av_N \left(\left\langle\!\!\left\langle \left| \sum_{i=1}^{r} K_i \right|^k \right\rangle\!\!\right\rangle_{N, \lambda K} \right) \\ &\leqslant k! \, r^k Av_N(\langle |K|^k \rangle_{N, \lambda K}) \end{aligned}$$

Since K can be arbitrarily large, we face here a minor version of the "large field problem." However, it can be resolved by the following estimate,

$$\begin{aligned} \frac{1}{k!} Av_N(\langle |K|^k \rangle_{N, \lambda K}) &\leqslant 2 Av_N(\langle e^K \rangle_{N, \lambda K}) \\ &= 2 Av_N \left(\frac{\langle e^{(\lambda+1)K} \rangle_N}{\langle e^{\lambda K} \rangle_N} \right) \\ &\leqslant 2 (Av_N[\langle e^{(\lambda+1)K} \rangle_N^2]^2)^{1/2} \times (Av_N[\langle e^{\lambda K} \rangle_N]^{-2})^{1/2} \end{aligned}$$

For the first vector, an elementary calculation gives the bound $e^{(1+\lambda)^2}$. For the second factor we get

$$Av_N[\langle e^{\lambda K} \rangle_N]^{-2} \leqslant Av_N(\exp[-2\lambda \langle K \rangle_N]) \leqslant e^{2\lambda^2} \tag{A.5}$$

wher we used first the Jensen inequality and then the fact that $\langle K \rangle$ is a gaussian variable of covariance $\leqslant 1$. ▮

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